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Symmetry representations in the rigged Hilbert space formulation of quantum mechanics

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Abstract

We discuss some basic properties of Lie group representations in rigged Hilbert spaces. In particular, we show that a differentiable representation in a rigged Hilbert space may be obtained as the projective limit of a family of continuous representations in a nested scale of Hilbert spaces. We also construct a couple of examples illustrative of the key features of group representations in rigged Hilbert spaces. Finally, we establish a simple criterion for the integrability of an operator Lie algebra in a rigged Hilbert space.

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1. Introduction

In this paper we undertake a study of differentiable representations of finite dimensional Lie groups in rigged Hilbert spaces (RHS). Since symmetry transformations on physical systems often constitute such Lie groups, these representations may prove to be an integral component of the relatively new rigged Hilbert space formulation of quantum physics [1–5]. The inceptive motivation for introducing RHS in quantum mechanics, especially in [1–3], was to provide Dirac's bra and ket formalism, already a well established calculational tool, with a proper mathematical content. It was later realized [4–7] that the mathematical structure of RHS contains a certain suppleness that is well suited for a systematic study of scattering and decay phenomena. During about the past two decades, investigations have continued into various aspects of the quantum theory of scattering and decay in the framework of RHS. Perhaps the most significant of these developments is the finding that in a suitably constructed RHS, the fundamental dynamical equation of Schrödinger $i\hbar \frac{\partial \psi}{\partial t} = H\psi$ can be integrated to obtain a Hamiltonian generated *semigroup* for the time evolution of the physical system [4, 5, 8]. This and certain other features of the theory show that the RHS formulation of quantum physics deviates from the orthodox Hilbert space theory in significant ways. They are also indicative of the above mentioned flexibility of the structure of RHS mathematics.

However, although the semigroup time evolution in RHS has been studied extensively [8] and often emphasized, a systematic study of representations of Lie groups in RHS has not been carried out in a general setting. Certain fundamental properties such representations must possess, as well as their physical content, have been discussed in [3, 10]. Even in these works, some of the most natural questions to address, such as obtaining an RHS representation of a Lie group from a given Hilbert space representation and/or from a given Lie algebra representation, have not been undertaken.

Apart from the rather obvious need as a component of the general RHS formulation of quantum mechanics, such a study of Lie group representations in RHS is also motivated by certain recent applications of the formalism to relativistic resonances and unstable particles [9]. These works develop a characterization of relativistic resonances and unstable particles by way of certain representations of a particular subsemigroup of the Poincaré group. The relevant subsemigroup, named the Poincaré semigroup [9], is in fact the semidirect product of the homogeneous Lorentz group with the *semigroup* of space–time translations into the forward light cone. The RHS representations of this subsemigroup can be characterized by a spin value j and a complex square mass value S_R , and consequently they can be attributed a physical interpretation as representing resonances along the lines of Wigner’s classic theory of the unitary representations of the Poincaré group for stable particles. These RHS representations of the Poincaré semigroup have subtleties which are not present in either the unitary representations in Hilbert spaces or the well understood RHS theory for the non-relativistic case [8] where only a one-parameter semigroup is needed to describe the evolution of the physical system. Many of the technical and theoretical issues appertaining to the RHS representations of the multi-parameter Poincaré semigroup [9] are subsumed under the subject of this paper. In the remainder of this introductory section we shall briefly state the questions that we attempt to formulate and answer in this paper; in sections 2 and 3 we present our results.

Definition 1.1. *A rigged Hilbert space consists of a triad of vector spaces*

$$\Phi \subset \mathcal{H} \subset \Phi^\times \tag{1.1}$$

where

1. \mathcal{H} is a Hilbert space
2. Φ is a dense subspace of \mathcal{H} and it is endowed with a complete locally convex topology τ_Φ that is stronger than the \mathcal{H} -topology
3. Φ^\times is the space of continuous antilinear functionals on Φ . It is complete in its weak* topology τ^\times and it contains \mathcal{H} as a dense subspace.

It is preceptive that the topology of the space Φ be constructed so as to yield an algebra \mathcal{A} of quantum physical observables—defined at the outset as an algebra of endomorphisms on a dense subspace \mathcal{D} of \mathcal{H} —continuous as mappings on Φ . For an operator A of this algebra (that is also self-adjoint, normal or unitary as an operator in \mathcal{H}), the nuclear spectral theorem of Gel’fand affirms the existence of generalized eigenvectors (i.e., eigenvectors of the dual operator A^\times in Φ^\times) with the corresponding eigenvalues ranging over the continuous (Hilbert space) spectrum of A ¹.

Thus, with the aid of RHS, the continuous and point spectra of observables can be treated on an equal footing. Further, the above set of eigenvectors constitute a basis for the space Φ .

¹ In Gel’fand’s proof of the theorem, the locally convex space Φ of definition 1.1 was required to be nuclear. Therefore, rigged Hilbert spaces are customarily defined in quantum theory with the requirement that Φ be nuclear. However, since this condition can be relaxed [3] and since the nuclearity of Φ is not needed for the purposes of this paper (and thus our results have a slightly broader generality), we choose to define RHSs as in definition 1.1, without demanding that Φ be nuclear (see also [11]).

This is in fact the mathematical content of Dirac's bra-and-ket formulation of quantum mechanics.

Very often in practice, the above mentioned algebra of observables \mathcal{A} (to be made continuous on Φ) arises as the associative algebra of an operator Lie algebra in \mathcal{H} . Further, this Lie algebra may be the differential $d\mathcal{T}$ (with respect to the norm topology of \mathcal{H}) of a continuous (often unitary) representation \mathcal{T} in \mathcal{H} of a Lie group G . As stated above, the complete locally convex space Φ for an RHS may be constructed from an invariant dense domain \mathcal{D} for the associative algebra of $d\mathcal{T}$ so that every element of this algebra becomes continuous as a mapping on Φ .

We prove (proposition 2.1) that the natural question whether the Hilbert space representation \mathcal{T} (say, when restricted to Φ) yields a representation of the group G in Φ is answerable in the affirmative, provided the invariant domain for the operator Lie algebra $d\mathcal{T}$ is chosen so that it remains invariant also under the group representation \mathcal{T} . Observe that this is a natural and minimal requirement for a homomorphism to be defined on G by the composition of the operators $\mathcal{T}|_{\Phi}$ which denote the restriction of \mathcal{T} to Φ . Moreover, it will be seen that the τ_{Φ} -generators of the representation $\mathcal{T}|_{\Phi}$ coincide with the $\tau_{\mathcal{H}}$ -generators of \mathcal{T} on the space Φ .

In contrast, it may also be possible to construct the space Φ from a dense domain \mathcal{D} which remains invariant under the differential $d\mathcal{T}$ but not under the group representation \mathcal{T} . This leads to the interesting possibility that certain symmetries present in the Hilbert space description of a quantum mechanical system need not be present in its RHS description. It is this feature that has been exploited in the above mentioned RHS study of certain quantum mechanical processes such as resonance scattering and decay, and in particular, the apparent asymmetric, semigroup time evolution associated with these processes. However, we shall not be concerned with these aspects of the RHS quantum theory in this paper.

Section 3 of this paper deals with the complementary question whether every (differentiable) Lie group representation in the space Φ of an RHS is necessarily obtained from a (continuous) representation of the group in the central Hilbert space \mathcal{H} . The starting point in this case is a representation T of a certain Lie algebra \mathcal{G} in a Hilbert space \mathcal{H} . Unlike in section 2, proposition 2.1, we will no longer assume that T is the differential $d\mathcal{T}$ of a continuous group representation \mathcal{T} in \mathcal{H} . Instead, we will establish a simple criterion of determining if the given Lie algebra representation T is the differential of a certain Lie group representation in Φ .

2. Induction from Hilbert space representations

Definition 2.1. A continuous representation of a Lie group G on a topological vector space Ψ is a continuous mapping $\mathcal{T}: G \times \Psi \rightarrow \Psi$, such that

1. for every $g \in G$, $\mathcal{T}(g)$ is a linear operator in Ψ
2. for every $\psi \in \Psi$ and $g_1, g_2 \in G$, $\mathcal{T}(g_1 g_2)\psi = \mathcal{T}(g_1)\mathcal{T}(g_2)\psi$

Definition 2.2. A differentiable representation of a Lie group G on a complete topological vector space Ψ is a mapping $\mathcal{T}: G \times \Psi \rightarrow \Psi$ which fulfils all the requirements of definition 2.1 and has the additional property that for any one-parameter subgroup $\{g(t)\}$ of G , $\lim_{t \rightarrow 0} \frac{\mathcal{T}(g(t))\phi - \phi}{t}$ exists for all $\phi \in \Psi$ (and, a fortiori, defines a continuous linear operator on Ψ).

Definition 2.3. A continuous one-parameter group of operators $\mathcal{T}(t)$ in a locally convex topological vector space Ψ is said to be equicontinuous if for every continuous seminorm p on Ψ , there exists another, q , such that

$$p(\mathcal{T}(t)\phi) \leq q(\phi) \tag{2.1}$$

holds for all $\phi \in \Psi$ and all $t \in \mathbb{R}$.

The one-parameter group is said to be locally equicontinuous if (2.1) holds for all t in every compact subset of \mathbb{R} .

Let G be a Lie group of dimension $d < \infty$, and \mathcal{G} be its Lie algebra. Let \mathcal{T} be a continuous representation of G in a Hilbert space \mathcal{H} , and let T be the differential of \mathcal{T} evaluated at the identity e of G , $d\mathcal{T}|_e = T$. It is well known that T furnishes a representation of \mathcal{G} by (not necessarily continuous) linear operators in \mathcal{H} .

Proposition 2.1. *Let G , \mathcal{G} , \mathcal{T} and T be as above. Let \mathcal{D} be a dense subspace of \mathcal{H} which remains invariant under both \mathcal{T} and T . Then there exists a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$ such that the restrictions $\mathcal{T}|_\Phi$ yield a continuous representation of G in Φ .*

Furthermore, if \mathcal{D} can be chosen so that it is complete under the projective topology τ_Φ ((2.3) below), the representation $\mathcal{T}|_\Phi$ of G is differentiable in Φ . By duality, there also exists a differentiable representation of G in Φ^\times .

Proof. Let $\{x_i\}_{i=1}^d$ be a basis for \mathcal{G} and let X_i be the restriction of the differential $T(x_i)$ to the invariant domain \mathcal{D} .

Construction of RHS

Define a family of scalar products on \mathcal{D} by setting

$$(\phi, \psi)_{n+1} = \sum_{i=1}^d (X_i \phi, X_i \psi)_n + (\phi, \psi)_n \quad n = 0, 1, 2, \dots, \quad \phi, \psi \in \mathcal{D} \quad (2.2)$$

where $(\phi, \psi)_0 \equiv (\phi, \psi)$, the scalar product which \mathcal{D} inherits from \mathcal{H} . Linearity of the X_i then ensures that $(\phi, \psi)_n$ is in fact a scalar product on \mathcal{D} for every n .

With (2.2), we have on \mathcal{D} the family of norms:

$$\|\phi\|_{n+1}^2 = \sum_{i=1}^d \|X_i \phi\|_n^2 + \|\phi\|_n^2. \quad (2.3)$$

From (2.3), it is clear that

$$\|\phi\|_n \leq \|\phi\|_{n+1} \quad \text{and} \quad \|X_i \phi\|_n \leq \|\phi\|_{n+1}. \quad (2.4)$$

Since the norms (2.3) are derived from the scalar products (2.2), the dense subspace \mathcal{D} can be completed with respect to each norm $\|\cdot\|_n$ to obtain a Hilbert space \mathcal{H}_n . The relations (2.4) then imply that the \mathcal{H}_n form a nested scale

$$\mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2 \supset \dots \quad (2.5)$$

and that the operators X_i , and therewith the algebra \mathcal{A} spanned by them, extend to elements of $\mathcal{B}(\mathcal{H}_{n+1}, \mathcal{H}_n)$, the space of bounded linear operators from \mathcal{H}_{n+1} into \mathcal{H}_n .

Now, let Φ be defined by

$$\Phi = \bigcap_n \mathcal{H}_n. \quad (2.6)$$

It is clear that Φ is a Fréchet space² which contains \mathcal{D} . It is also easy to see that the topology of Φ is independent of the basis chosen. Φ is dense in \mathcal{H} , and thus we have the triplet

$$\Phi \subset \mathcal{H} \subset \Phi^\times \quad (2.7)$$

² The topology of Φ induced by the countable family of norms (2.3) is equivalent to the topology induced by the powers of the generalized Laplacian $(\sum_{i=1}^d X_i^2 + I)^n$, as considered in [12].

where Φ^\times , the anti-dual of Φ , can be obtained as

$$\Phi^\times = \bigcup_n \mathcal{H}_n. \quad (2.8)$$

Remark: It is not known to us if the space Φ is nuclear when it is constructed in the above manner, i.e., under the projective topology from the differential of a continuous representation of a finite dimensional but otherwise arbitrary Lie group in a Hilbert space. However, it is known that nuclearity holds for τ_Φ for the unitary representations of the following classes of Lie groups: semi-simple groups [12]; nilpotent groups [13]; semi-direct products of Abelian groups with compact groups [13] and the Poincaré group. Thus for a large class of Lie groups, our proposition 2.1 can be restated for a triad $\Phi \subset \mathcal{H} \subset \Phi^\times$, where Φ is a nuclear space.

Restriction of \mathcal{T} to Φ

From the \mathcal{H} -continuity of $\mathcal{T}(g)$, we have, for all $\psi \in \mathcal{H}$,

$$\|\mathcal{T}(g)\psi\| \leq \omega(g)\|\psi\| \quad (2.9)$$

where $\omega(g)$ is a positive constant which may depend on the group element g . An important property of the representation \mathcal{T} is that it is locally equicontinuous, a consequence of the local equicontinuity of continuous, one-parameter groups in barrelled spaces [14]. That is, the positive valued function ω on G is locally bounded.

Proposition 2.1 follows from (2.9) and the following operator valued formulation of the well known Lie algebra inner automorphism $\text{Ad}(e^{ty})$ of \mathcal{G} , defined by $z \rightarrow e^{ty} z e^{-ty}$, $y, z \in \mathcal{G}$ (in any realization). Thus, for $g = e^y$,

$$gzg^{-1} = e^{(\text{ady})z} \equiv f_{zi}(g^{-1})x_i \quad (2.10)$$

where the functions f_{zi} are locally analytic on G . The corresponding automorphism on G is $g e^{tz} g^{-1} = e^{t(\exp(\text{ady})z)}$, where $g = e^y$ and t , a real parameter. Then, for $\phi \in \mathcal{D}$,

$$\frac{d}{dt} \mathcal{T}(g) \mathcal{T}(e^{tz}) \mathcal{T}(g^{-1}) \phi = \frac{d}{dt} \mathcal{T}(e^{t(\exp(\text{ady})z)}) \phi. \quad (2.11)$$

Now, since

$$\begin{aligned} \lim_{t \rightarrow 0} \left\| \frac{\mathcal{T}(g) \mathcal{T}(e^{tz}) \mathcal{T}(g^{-1}) \phi - \phi}{t} - \mathcal{T}(g) \mathcal{T}(z) \mathcal{T}(g^{-1}) \phi \right\| \\ \leq \omega(g) \lim_{t \rightarrow 0} \left\| \left(\frac{\mathcal{T}(e^{tz}) - I}{t} - \mathcal{T}(z) \right) \mathcal{T}(g^{-1}) \phi \right\| \end{aligned} \quad (2.12)$$

and \mathcal{D} is invariant under \mathcal{T} , we see that the left-hand side of (2.11), evaluated at $t = 0$, is $\mathcal{T}(g) \mathcal{T}(z) \mathcal{T}(g^{-1}) \phi$. Thus,

$$\mathcal{T}(g) \mathcal{T}(z) \mathcal{T}(g^{-1}) \phi = \mathcal{T}((e^{\text{ady}})z) \phi \quad (2.13)$$

for $\phi \in \mathcal{D}$. But, by (2.10), for the basis elements X_i we then have

$$\mathcal{T}(g) X_i \mathcal{T}(g^{-1}) \phi = \sum_{j=1}^d f_{ij}(g^{-1}) X_j \phi. \quad (2.14)$$

The real valued functions f_{ij} are continuous and locally analytic, and provide a (not necessarily faithful) matrix representation of G . For the one-parameter subgroup $\{e^{tx_k}\}$, it is easy to see that the f_{ij} can be expanded as

$$f_{ij}(e^{-tx_k}) = \delta_{ij} + t c_{ijk} + \dots \quad (2.15)$$

where c_{ijk} are the structure constants of \mathcal{G} . Furthermore, the f_{ij} and c_{ijk} fulfil the identities

$$\sum_k c_{ijk} f_{kl}(g^{-1}) = \sum_{m,n} c_{mnl} f_{im}(g^{-1}) f_{jn}(g^{-1}). \quad (2.16)$$

The relations (2.9) and (2.14) show that for any $\phi \in \mathcal{D}$,

$$\|\mathcal{T}(g)\phi\|_n \leq \omega(g) \left(1 + \sum_{i,j=1}^d |f_{ij}(g)| \right)^n \|\phi\|_n. \quad (2.17)$$

The proof of (2.17) is by induction. For $n = 0$, (2.17) is just (2.9), the assumed continuity of \mathcal{T} in \mathcal{H} . If (2.17) holds for some n , then,

$$\begin{aligned} \|\mathcal{T}(g)\phi\|_{n+1}^2 &= \sum_{i=1}^d \|X_i \mathcal{T}(g)\phi\|_n^2 + \|\mathcal{T}(g)\phi\|_n^2 \\ &= \sum_{i=1}^d \|\mathcal{T}(g)\mathcal{T}(g^{-1})X_i \mathcal{T}(g)\phi\|_n^2 + \|\mathcal{T}(g)\phi\|_n^2 \\ &\leq \omega(g)^2 \left(1 + \sum_{i,j=1}^d |f_{ij}(g)| \right)^{2n} \left(\sum_{k=1}^d \|\mathcal{T}(g^{-1})X_k \mathcal{T}(g)\phi\|_n^2 + \|\phi\|_n^2 \right) \\ &\leq \omega(g)^2 \left(1 + \sum_{i,j=1}^d |f_{ij}(g)| \right)^{2n} \left(1 + \sum_{k,l=1}^d |f_{kl}(g)| \right)^2 \|\phi\|_{n+1}^2 \\ &\leq \omega(g)^2 \left(1 + \sum_{i,j=1}^d |f_{ij}(g)| \right)^{2n+2} \|\phi\|_{n+1}^2 \end{aligned} \quad (2.18)$$

where the inequalities (2.4) are used in the last step. Thus, we have (2.17).

The relation (2.17) gives the continuity of the operators $\mathcal{T}(g)$, $g \in G$, (when restricted to the dense domain \mathcal{D}) with respect to the Fréchet topology given by (2.2) or (2.3). It is also fairly straightforward to establish the continuity of the mapping $G \rightarrow \mathcal{T}(G)$ in this topology on \mathcal{D} . To that end, for $\phi \in \mathcal{D}$,

$$\|\mathcal{T}(g)\phi - \phi\|_{n+1}^2 = \sum_{i=1}^d \|X_i \mathcal{T}(g)\phi - X_i \phi\|_n^2 + \|\mathcal{T}(g)\phi - \phi\|_n^2. \quad (2.19)$$

Then, since

$$\begin{aligned} \|X_i \mathcal{T}(g)\phi - X_i \phi\|_n &= \left\| \mathcal{T}(g) \sum_{i,j=1}^d f_{ij}(g) X_j \phi - X_i \phi \right\|_n \\ &\leq \omega(g) \left(1 + \left| \sum_{i,j=1}^d f_{ij}(g) \right| \right)^n \left\| \sum_{i,j=1}^d f_{ij}(g) X_j \phi - X_i \phi \right\|_n + \|\mathcal{T}(g)X_i \phi - X_i \phi\|_n \end{aligned} \quad (2.20)$$

and since from (2.15), $\lim_{g \rightarrow e} f_{ij}(g^{-1}) = \delta_{ij}$, the continuity $\lim_{t \rightarrow 0} \|\mathcal{T}(e^{tx})\phi - \phi\|_n = 0$ implies

$$\lim_{t \rightarrow 0} \|\mathcal{T}(e^{tx})\phi - \phi\|_{n+1} = 0. \quad (2.21)$$

Since \mathcal{D} is dense in each Hilbert space \mathcal{H}_n of the nested scale (2.5), linearity of the operators $\mathcal{T}(g)$ permits the inequalities (2.17) and (2.21) to be extended to the whole of \mathcal{H}_n . That is, the representation $\mathcal{T}|_{\mathcal{D}}$ extends from \mathcal{D} to a continuous representation of G in each of the Hilbert spaces \mathcal{H}_n of (2.5).

Since $\Phi = \bigcap_{n=0}^{\infty} \mathcal{H}_n$, the relations (2.17) and (2.21) can be extended to the space Φ . Therewith we conclude that the restrictions $\mathcal{T}(g)|_{\Phi}$ to the space Φ yield a continuous (with respect to the Φ -topology (2.3)) representation of G on Φ .

It remains to prove that this representation on Φ is differentiable, i.e., for any $\phi \in \Phi$ and $x \in \mathcal{G}$, $\lim_{t \rightarrow 0} \frac{\mathcal{T}(e^{tx}) - I}{t} \phi$ exists. We shall shortly see that the equality

$$\lim_{t \rightarrow 0} \left\| \frac{\mathcal{T}(e^{tx}) - I}{t} \phi - \mathcal{T}(x)\phi \right\|_n = 0 \quad (2.22)$$

can be easily obtained by induction so long as ϕ is restricted to the dense domain \mathcal{D} . However, since the mapping $G \rightarrow \mathcal{T}(G)$ is not linear, we cannot necessarily extend (2.22) to the whole of Φ .

At this point we remark that a result of Roberts, proposition 13 in [1], leads to the conclusion that the invariant domain \mathcal{D} is complete under the projective topology when \mathcal{D} is taken to be the maximal invariant domain for the operator Lie algebra $\mathcal{T}(\mathcal{G})$. This domain is also invariant under the operator group $\mathcal{T}(G)$. Thus, for such \mathcal{D} , (2.22) holds for all $\phi \in \Phi$, and we have a differentiable representation of G on Φ .

To prove (2.22), notice first that for $n = 0$ the equation just expresses that differentiability of ϕ in \mathcal{H} -topology, and thus the equation is true for all $\phi \in \mathcal{D}$ by the definition of \mathcal{D} . Next, if (2.22) is true for some n , then

$$\begin{aligned} \lim_{t \rightarrow 0} \left\| \frac{\mathcal{T}(e^{tx_i}) - I}{t} \phi - \mathcal{T}(x_i)\phi \right\|_{n+1}^2 &= \lim_{t \rightarrow 0} \left(\sum_{j=1}^d \left\| X_j \left(\frac{\mathcal{T}(e^{tx_i}) - I}{t} \phi - X_i \phi \right) \right\|_n^2 \right. \\ &\quad \left. + \left\| \frac{\mathcal{T}(e^{tx_i}) - I}{t} \phi - X_i \phi \right\|_n^2 \right). \end{aligned} \quad (2.23)$$

Since (2.22) is assumed to be true for n , the last term vanishes. Also,

$$\begin{aligned} \left\| X_j \left(\frac{\mathcal{T}(e^{tx_i}) - I}{t} - X_i \right) \phi \right\|_n &= \left\| \frac{\sum_k f_{jk}(e^{tx_i}) \mathcal{T}(e^{tx_i}) X_k \phi - X_j \phi}{t} - X_j X_i \phi \right\|_n \\ &\leq \left\| \frac{f_{jj}(e^{tx_i}) \mathcal{T}(e^{tx_i}) X_j \phi - X_j \phi}{t} - X_i X_j \phi \right\|_n \\ &\quad + \left\| \frac{\sum_{k \neq j} (f_{jk}(e^{tx_i}) \mathcal{T}(e^{tx_i}) X_k \phi - t c_{jik} X_k \phi)}{t} \right\|_n. \end{aligned} \quad (2.24)$$

The invariance of \mathcal{D} under X_k and the expansion (2.15) of the f_{ij} show that the right-hand side of (2.24) vanishes when $t \rightarrow 0$. That is, the right-hand side of (2.23) tends to zero. This proves (2.22) for every basis element x_i of \mathcal{G} . The general case easily follows.

The existence of a differentiable representation of G in Φ^\times easily follows from the treatment in section 2.3.

This concludes the proof of proposition 2.1. \square

Proposition 2.1 thus shows that, starting from a continuous representation of a finite dimensional Lie group in a Hilbert space \mathcal{H} , a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$ can be constructed so that there exists a differentiable representation of the group in Φ . The construction begins with identifying the maximal invariant domain for the operator Lie algebra

in \mathcal{H} . In view of the remark following (2.8), for unitary representations of a large class of Lie groups we can construct the triad $\Phi \subset \mathcal{H} \subset \Phi^\times$ subject to the more restrictive condition that Φ be a nuclear space.

In the remainder of this section we shall investigate some secondary aspects of such representations in Φ and present a couple of simple examples illustrating these features.

2.1. One-parameter subgroups in Φ

Proposition 2.1 asserts that the differentiable representation \mathcal{T}_Φ of a finite dimensional Lie group G , obtained from its continuous Hilbert space representation \mathcal{T} , is precisely the projective limit of a family of continuous representations in the nested scale of Hilbert spaces \mathcal{H}_n in (2.5). That is, the representation \mathcal{T}_Φ in Φ extends to a continuous representation \mathcal{T}_n ($\mathcal{T}_0 = \mathcal{T}$) of G in \mathcal{H}_n for $n = 0, 1, 2, \dots$. The generators $X_{i,n}$ of the one-parameter subgroups $\mathcal{T}_n(e^{tX_i})$ are the extensions to \mathcal{H}_n by closure, with respect to the norm topology $\|\cdot\|_n$, of the operators X_i in Φ , and they furnish a representation of the Lie algebra \mathcal{G} in some algebra $\mathcal{A}(\mathcal{D}_n)$ of endomorphisms on a dense subspace \mathcal{D}_n of \mathcal{H}_n . In fact, the invariant subspace \mathcal{D} from which the Fréchet space Φ was constructed can serve as \mathcal{D}_n in each \mathcal{H}_n .

This observation motivates us to consider the problem of integrating the Lie algebra representation $T(\mathcal{G})$ in Φ to the differentiable group representation \mathcal{T}_Φ as, somewhat loosely put, the projective limit of the integrability problem in the Hilbert spaces \mathcal{H}_n . We shall take up this integrability of an operator Lie algebra in Φ as a substantive problem below in section 3. Here we will limit ourselves to the integrability conditions on a single element of $T(\mathcal{G})$ into a differentiable one-parameter group in Φ . More precisely, the integrability of an element X of the continuous Lie algebra representation T in Φ to a differentiable one-parameter group can be treated as a repeated application of the classical Hille–Yosida [17, 18] theory of one-parameter C_0 -groups in Banach spaces.

Consider again the case studied in proposition 2.1. Let us denote a typical one-parameter subgroup of this differentiable representation by $\mathcal{T}_\Phi(t, X)$, where X is the generator of $\mathcal{T}_\Phi(t, X)$. As seen from (2.17), the differentiable subgroup $\mathcal{T}_\Phi(t, X)$ extends to a C_0 -group [17, 18] in each of the Hilbert spaces \mathcal{H}_n . In \mathcal{H}_n , this subgroup is generated by \bar{X}_n , the extension to \mathcal{H}_n , by closure, of the operator X in Φ . If we denote this C_0 -group in \mathcal{H}_n by $\mathcal{T}(t, \bar{X}_n)$, then $\mathcal{T}_\Phi(t, X)$ in Φ is the projective limit of the C_0 -groups $\mathcal{T}(t, \bar{X}_n)$ in \mathcal{H}_n .

Suppose $\mathcal{T}(t, \bar{X}_n)$ is of type ω_n [17, 18], i.e.,

$$\omega_n = \inf_{t \neq 0} \frac{1}{|t|} \ln \|\mathcal{T}(t, \bar{X}_n)\|_n = \pm \lim_{t \rightarrow \pm\infty} \frac{1}{|t|} \ln \|\mathcal{T}(t, \bar{X}_n)\|_n. \quad (2.25)$$

The classical Hille–Yosida theory affirms the following relationships between the resolvent $R(\lambda, \bar{X}_n)$ of \bar{X}_n and the C_0 -group $\mathcal{T}(t, \bar{X}_n)$ generated by \bar{X}_n :

$$R(\lambda, \bar{X}_n)\phi = \int_0^\infty dt e^{-\lambda t} \mathcal{T}(t, \bar{X}_n)\phi \quad \lambda > \omega_n \quad (2.26)$$

$$R(\lambda, \bar{X}_n)\phi = - \int_{-\infty}^0 dt e^{-\lambda t} \mathcal{T}(t, \bar{X}_n)\phi \quad \lambda < -\omega_n$$

$$\mathcal{T}(t, \bar{X}_n)\phi = \lim_{\lambda \rightarrow \infty} e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} (\lambda R(\lambda, \bar{X}_n))^j \phi \quad t > 0 \quad (2.27)$$

$$\mathcal{T}(t, \bar{X}_n)\phi = \lim_{\lambda \rightarrow -\infty} e^{-\lambda t} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} (\lambda R(\lambda, \bar{X}_n))^j \phi \quad t < 0$$

where all limits are with respect to the \mathcal{H}_n -topology. Further, for some positive M_n and $\beta_n > \omega_n$, we have

$$\|(R(\lambda, \bar{X}_n))^p\|_n \leq M_n(|\lambda| - \beta_n)^{-p} \quad (2.28)$$

for all $\lambda > \beta_n$ and $p = 1, 2, 3, \dots$. In fact, the relation (2.28) is a necessary and sufficient requirement for the closed operator \bar{X}_n to generate the C_0 -group $\mathcal{T}(t, \bar{X}_n)$ in the Hilbert space \mathcal{H}_n .

Since the differentiable subgroup $\mathcal{T}_\Phi(t, X)$ in Φ is the projective limit of the continuous groups $\mathcal{T}_\Phi(t, \bar{X}_n)$, we see that the continuous operator X generates a one-parameter group in Φ when its closure \bar{X}_n fulfils the relation (2.28) for all $n = 0, 1, 2, \dots$. That is, for the kind of differentiable subgroup considered here, the problem of reconstructing the $\mathcal{T}_\Phi(t, X)$ in terms of (the resolvent of) X in Φ can be reduced to the corresponding problem in each of the \mathcal{H}_n in the nested scale of Hilbert spaces (2.5).

It is interesting at this point to ask if the subgroup $\mathcal{T}_\Phi(t, X)$ can be recovered from its generator X in Φ without appealing to the Banach space theory applied to the Hilbert spaces \mathcal{H}_n . The theory of C_0 -groups in more general locally convex spaces has also been developed [18], and the form of this general theory is similar to the Banach space theory when the group is equicontinuous in the parameter. For such a C_0 -group in a locally convex space, the resolvent operator of the generator can be obtained in much the same way as in (2.26) as the Laplace transform of the group. The group, in turn, can be recovered from the resolvent by way of a limiting process similar to (2.27). Of course the integrals and limit processes are now to be defined with respect to the locally convex topology of the vector space.

Nevertheless, as evident from the example below, such global equicontinuity may prove to be too strong a restriction for C_0 -groups in rigged Hilbert spaces. In such situations, the resolvent operator $R(\lambda, X)$ may fail to exist anywhere in the complex plane, and further, even when it does exist for all large $|\lambda|$, the group may not be able to be constructed from it as in (2.27)³.

One obvious condition under which the resolvent operator $R(\lambda, X)$ can acquire an integral resolution of the kind (2.26) in Φ is

$$\omega \equiv \sup_n \omega_n < \infty \quad (2.29)$$

where the ω_n are defined as in (2.25) and $|\lambda| > |\omega|$. However, even when the resolvent $R(\lambda, X)$ of X is everywhere defined in the complex plane, it is not necessary that the subgroup $\mathcal{T}_\Phi(t, X)$ can be recovered in terms of $R(\lambda, X)$ by the limit process (2.27) (in the Φ -topology). One instance when this is possible is

$$M_n \leq 1 \quad \text{and} \quad \beta \equiv \sup_n \beta_n < \infty \quad (2.30)$$

where M_n and β_n are defined as in (2.28). This condition assures that the Hille–Yosida theory for the C_0 -groups in locally convex spaces [18] can be applied. In other words, if the relations (2.29) and (2.30) hold, the subgroup $\mathcal{T}_\Phi(t, X)$ can be recovered from the resolvent of its generator by way of (2.27), defined now in Φ as a τ_Φ -limit process.

2.2. Example

Define a multiplication in \mathbb{R}^3 by

$$(\xi_1, \xi_2, \xi_3)(\zeta_1, \zeta_2, \zeta_3) = (\xi_1 + \zeta_1, \xi_2 + \zeta_2, \xi_3 + \zeta_3 + \xi_1 \zeta_2). \quad (2.31)$$

³ As remarked earlier, one-parameter C_0 -groups in Φ are necessarily locally equicontinuous, and these groups have been studied in the literature [14]. However, we shall not make use of the results of [14] as the structure of Φ , defined by (2.6), makes the case considerably simpler for one-parameter groups in rigged Hilbert spaces.

Under this multiplication \mathbb{R}^3 becomes a group, G , which has the set $\{(0, 0, \xi_3)\}$ as its centre. The Lie algebra \mathcal{G} of G is spanned by the elements

$$\chi_1 = (1, 0, 0) \quad \chi_2 = (0, 1, 0) \quad \chi_3 = (0, 0, 1) \quad (2.32)$$

which fulfil the commutation relations

$$[\chi_1, \chi_2] = \chi_3 \quad [\chi_1, \chi_3] = [\chi_2, \chi_3] = 0. \quad (2.33)$$

These commutation relations can be realized in \mathbb{R}^3 by the multiplication rule defined, for any two elements $\chi = (\alpha, \beta, \gamma)$ and $\chi' = (a, b, c)$ of \mathcal{G} , as

$$(\alpha, \beta, \gamma)(a, b, c) = (0, 0, \alpha b). \quad (2.34)$$

Thus, the basis elements (2.32) fulfil the relations

$$\chi_i \chi_j = \delta_{ij} \delta_{2j} \chi_3. \quad (2.35)$$

Notice that under the product rule (2.34), the Lie algebra \mathcal{G} becomes an associative algebra. This associative algebra can be made into an operator algebra on \mathbb{R}^3 by way of the definition, for $\chi = (\alpha, \beta, \gamma) \in \mathcal{G}$ and $v = (x, y, z) \in \mathbb{R}^3$:

$$\chi v = (\alpha y + \gamma z, \beta z, 0). \quad (2.36)$$

The group G can be constructed by the exponentiation of \mathcal{G} :

$$(\xi_1, \xi_2, \xi_3) = e + \xi_1 \chi_1 + \xi_2 \chi_2 + \xi_3 \chi_3 \quad (2.37)$$

where e , the identity element of G , is simply the origin $(0, 0, 0)$.

A representation \mathcal{T} of G in $L^2(\mathbb{R}, \mu)$, where μ is the Lebesgue measure, can be obtained by setting

$$(\mathcal{T}((\xi_1, \xi_2, \xi_3))f)(x) = e^{-i\xi_3} e^{-ix\xi_2} f(x + \xi_1). \quad (2.38)$$

It is easily seen that this is a continuous unitary representation of G .

The representation of \mathcal{G} , given by the differential $d\mathcal{T}$ (with respect to the L^2 -topology), is spanned by the operators

$$T(\chi_1) \equiv X_1 = \frac{d}{dx} \quad T(\chi_2) \equiv X_2 = -ix \quad T(\chi_3) \equiv X_3 = iI. \quad (2.39)$$

The task at hand is to construct a rigged Hilbert space so that a differentiable representation of G may be induced in the space Φ from the continuous unitary representation (2.38) in L^2 . To that end, as a common invariant domain for the operator Lie algebra (2.39) we choose the Schwartz space $\mathcal{S}(\mathbb{R})$, the space of C^∞ -functions which decay at infinity faster than the inverse of any polynomial. The definition (2.38) shows that $\mathcal{S}(\mathbb{R})$ is invariant under the group representation \mathcal{T} . We can now introduce the projective topology (2.3) on $\mathcal{S}(\mathbb{R})$ by means of the generators X_1, X_2 and X_3 of (2.39):

$$\|f\|_{n+1}^2 = \|X_1 f\|_n^2 + \|X_2 f\|_n^2 + \|f\|_n^2 \quad f \in \mathcal{S}(\mathbb{R}). \quad (2.40)$$

This topology on $\mathcal{S}(\mathbb{R})$ is equivalent to the more customary one defined by the norms $\|f\|_{m,n} = \sup_{x \in \mathbb{R}} |(\frac{d^m}{dx^m} x^n f)(x)|$. Thus, $\mathcal{S}(\mathbb{R})$ is complete under the topology (2.40) and, in fact, it is the projective limit of the scale of Hilbert spaces $L^2(\mathbb{R}, \mu) \supset \mathcal{H}_1 \supset \mathcal{H}_2 \cdots$ where \mathcal{H}_n is obtained by completing $\mathcal{S}(\mathbb{R})$ with respect to the norm $\|\cdot\|_n$. Therefore, we have the RHS

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}, \mu) \subset \mathcal{S}(\mathbb{R})^\times. \quad (2.41)$$

It is noteworthy that $\mathcal{S}(\mathbb{R})$ is a nuclear space.

Proposition 2.1 shows that the restriction of the continuous unitary representation (2.38) to the space $\mathcal{S}(\mathbb{R})$ yields therein a differentiable representation of the group (2.31). In fact, with respect to the norms (2.40),

$$\|\mathcal{T}(g)f\|_n \leq (1 + |\xi_1|^2 + |\xi_2|^2)^{n/2} \|f\|_n \quad n = 0, 1, 2, \dots, f \in \mathcal{S}(\mathbb{R}) \quad (2.42)$$

where $g = (\xi_1, \xi_2, \xi_3)$. Further,

$$\lim_{g \rightarrow e} \|(\mathcal{T}(g) - I)f\|_n = 0 \quad n = 0, 1, 2, \dots, f \in \mathcal{S}(\mathbb{R}) \quad (2.43)$$

and

$$\begin{aligned} \lim_{\xi_1 \rightarrow 0} \left\| \left(\frac{\mathcal{T}((\xi_1, 0, 0)) - I}{\xi_1} - X_1 \right) f \right\|_n &= 0 \\ \lim_{\xi_2 \rightarrow 0} \left\| \left(\frac{\mathcal{T}((0, \xi_2, 0)) - I}{\xi_2} - X_2 \right) f \right\|_n &= 0 \\ \lim_{\xi_3 \rightarrow 0} \left\| \left(\frac{\mathcal{T}((0, 0, \xi_3)) - I}{\xi_3} - X_3 \right) f \right\|_n &= 0 \quad n = 0, 1, 2, \dots, f \in \mathcal{S}(\mathbb{R}). \end{aligned} \quad (2.44)$$

As in the general case discussed in proposition 2.1, the proofs of (2.42)–(2.44) are by induction. The explicit form of the factor $(1 + |\xi_1|^2 + |\xi_2|^2)^{n/2}$ in (2.42) follows from that of the functions f_{ij} of (2.14), i.e., from $\mathcal{T}(g)X_1\mathcal{T}(g^{-1}) = X_1 + i\xi_2$, $\mathcal{T}(g)X_2\mathcal{T}(g^{-1}) = X_2 + i\xi_1$, or $f_{ij}(g) = \delta_{1i}\delta_{1j} + \delta_{2i}\delta_{2j} + \delta_{3i}\delta_{3j} + \xi_2\delta_{1i}\delta_{3j} + \xi_1\delta_{2i}\delta_{3j}$. In fact, the f 's are realized by the (non-isomorphic) representation $(\xi_1, \xi_2, \xi_3) \rightarrow (0, \xi_1, \xi_2)$ of G .

It is easily seen from (2.42) that the differentiable representation \mathcal{T} extends to a continuous representation \mathcal{T}_n for every n . The generators of the one-parameter subgroups $\mathcal{T}_n(\xi_1)$ and $\mathcal{T}_n(\xi_2)$ are, respectively, the extensions to \mathcal{H}_n , by closure, of X_1 and X_2 . As before, let us denote these two one-parameter subgroups in \mathcal{H}_n by $\mathcal{T}(\xi_1, \bar{X}_{1,n})$ and $\mathcal{T}(\xi_2, \bar{X}_{2,n})$. The classical Hille–Yosida theory can then be applied to recover these one-parameter subgroups from (the resolvents of) their generators.

Consider the one-parameter subgroup $\mathcal{T}(\xi_2, \bar{X}_{2,n})$. From (2.42),

$$\|\mathcal{T}(\xi_2, \bar{X}_{2,n})f\|_n \leq (1 + |\xi_2|^2)^{n/2} \|f\|_n \quad f \in \mathcal{H}_n. \quad (2.45)$$

It is of type $\omega_n = 0$, i.e.,

$$\omega_n = \inf_{|\xi_2|} \frac{1}{|\xi_2|} \ln \|\mathcal{T}(\xi_2, \bar{X}_{2,n})\|_n = 0 \quad n = 0, 1, 2, \dots \quad (2.46)$$

Then, the resolvent operator $R(\lambda, \bar{X}_{2,n})$ can be obtained as, for $f \in \mathcal{H}_n$,

$$\begin{aligned} R(\lambda, \bar{X}_{2,n})f &= \int_0^\infty d\xi_2 e^{-\lambda\xi_2} \mathcal{T}(\xi_2, \bar{X}_{2,n})f & \Re(\lambda) > 0 \\ R(\lambda, \bar{X}_{2,n})f &= - \int_{-\infty}^0 d\xi_2 e^{-\lambda\xi_2} \mathcal{T}(\xi_2, \bar{X}_{2,n})f & \Re(\lambda) < 0. \end{aligned} \quad (2.47)$$

Also, the $R(\lambda, \bar{X}_{2,n})$ satisfy the equicontinuity condition

$$\|(R(\lambda, \bar{X}_{2,n}))^p\|_n \leq (|\lambda| - n)^{-p} \quad (2.48)$$

for all λ with $|\Re(\lambda)| > n$. Therefore, according to the Hille–Yosida theory, the continuous group $\mathcal{T}(\xi_2, \bar{X}_{2,n})$ can be recovered from the resolvent $R(\lambda, \bar{X}_{2,n})$ by means of the limiting

process (2.27):

$$\begin{aligned} T(\xi_2, \bar{X}_n)\phi &= \lim_{\lambda \rightarrow \infty} e^{-\lambda \xi_2} \sum_{j=0}^{\infty} \frac{(\lambda \xi_2)^j}{j!} (\lambda R(\lambda, \bar{X}_n))^j \phi && \text{for } \xi_2 > 0 \\ T(\xi_2, \bar{X}_n)\phi &= \lim_{\lambda \rightarrow -\infty} e^{-\lambda \xi_2} \sum_{j=0}^{\infty} \frac{(\lambda \xi_2)^j}{j!} (\lambda R(\lambda, \bar{X}_n))^j \phi && \text{for } \xi_2 < 0. \end{aligned} \quad (2.49)$$

The differentiable one-parameter subgroup $T(\xi_2, X_2)$ in Φ can then be obtained as the projective limit of the continuous groups $T(\xi_2, \bar{X}_{2,n})$ in \mathcal{H}_n .

It is interesting to ask if the differentiable subgroup $T(\xi_2, X_2)$ can be recovered from the resolvent operator $R(\lambda, X_2)$ in Φ , i.e., without appealing to the Banach space theory applied to \mathcal{H}_n . First note that since

$$\omega = \sup_n \omega_n = 0 \quad (2.50)$$

where the ω_n are as in (2.46), the resolvent operator $R(\lambda, X_2)$ is defined everywhere on the complex plane, except on the imaginary axis, and it is given by integrals of the kind (2.47). The formal integrals

$$\begin{aligned} \int_0^{\infty} d\xi_2 e^{-\lambda \xi_2} e^{-ix \xi_2} f(x) &= \frac{1}{\lambda + ix} f(x) && \Re(\lambda) > 0 \\ - \int_{-\infty}^0 d\xi_2 e^{-\lambda \xi_2} e^{-ix \xi_2} f(x) &= \frac{1}{\lambda + ix} f(x) && \Re(\lambda) < 0 \end{aligned} \quad (2.51)$$

which must coincide with the vector valued ones (which exist by the above Hille–Yosida argument) show, for $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} R(\lambda, X_2)f(x) &= \frac{1}{\lambda + ix} f(x) = \int_0^{\infty} d\xi_2 e^{-\lambda \xi_2} T(\xi_2, X_2)f(x) && \Re(\lambda) > 0 \\ R(\lambda, X_2)f(x) &= \frac{1}{\lambda + ix} f(x) = - \int_{-\infty}^0 d\xi_2 e^{-\lambda \xi_2} T(\xi_2, X_2)f(x) && \Re(\lambda) < 0 \end{aligned} \quad (2.52)$$

where the integrals are defined as the limit of a Riemann sum with respect to the Fréchet topology (2.40) of $\mathcal{S}(\mathbb{R})$. The Hille–Yosida theory then implies that the operator $R(\lambda, X_2)$ is an everywhere defined continuous operator in $\mathcal{S}(\mathbb{R})$. Alternatively, we could directly show, by induction, that the linear operator defined by the first equality in (2.52) is such an operator:

$$\|R(\lambda, X_2)f\|_n \leq \left(\prod_{i=0}^n c_i \right)^{1/2} \|f\|_n \quad (2.53)$$

where $c_i = 1 + \prod_{j=0}^{i-1} c_j$, $i = 1, 2, \dots, n$, and $c_0 = \frac{1}{|\lambda|^2}$.

The relation (2.53) also shows that $R(\lambda, X_2)$ extends to an everywhere defined continuous operator in \mathcal{H}_n . This extension is really the resolvent operator $R(\lambda, \bar{X}_{2,n})$ of $\bar{X}_{2,n}$, the closure of X_2 in \mathcal{H}_n -topology. Further, a direct computation shows

$$\|(R(\lambda, \bar{X}_{2,n}))^p\|_n \leq (|\lambda| - n)^{-p} \quad (2.54)$$

for all $|\lambda| > n$ and $p = 1, 2, 3, \dots$. This is exactly the relation (2.48), obtained there by applying the Hille–Yosida theory to the C_0 -group $T(\xi_2, \bar{X}_{2,n})$ in \mathcal{H}_n .

The $\inf\{|\lambda|\}$, for which (2.54) holds, strictly increases along the scale $L^2(\mathbb{R}, \mu) \supset \mathcal{H}_1 \supset \mathcal{H}_2 \cdots$. This means that the upper bound (2.30) does not exist for the C_0 -group $T(\xi_2, X_2)$ in $\mathcal{S}(\mathbb{R})$. That is, there exist no $\beta \in \mathbb{R}$ such that $e^{-\beta \xi_2} T(\xi_2, X_2)$ is equicontinuous in $\mathcal{S}(\mathbb{R})$. Therefore, although the resolvent operator $R(\lambda, X_2)$ exists for all λ with $\Re(\lambda) \neq 0$, the

C_0 -group $\mathcal{T}(\xi_2, X_2)$ cannot be recovered from it by means of a limit process akin to (2.27) in the $\mathcal{S}(\mathbb{R})$ -topology. However, this recovery can be done for each $\mathcal{T}(\xi_2, \bar{X}_{2,n})$ in \mathcal{H}_n , and the differentiable group $\mathcal{T}(\xi_2, X_2)$ in $\mathcal{S}(\mathbb{R})$ can be obtained as the projective limit of the $\mathcal{T}(\xi_2, \bar{X}_{2,n})$ thus recovered.

2.3. Differentiable representations of groups in Φ^\times

Let \mathcal{T} be a representation of a finite dimensional Lie group G in the space Φ of a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$. Then, a representation \mathcal{V} of G can be defined in Φ^\times by way of the identity

$$\langle \mathcal{T}(g)\phi | F \rangle = \langle \phi | \mathcal{V}(g^{-1})F \rangle \quad g \in G \quad \phi \in \Phi \quad F \in \Phi^\times. \quad (2.55)$$

In other words,

$$\mathcal{V}(g^{-1}) = (\mathcal{T}(g))^\times \quad (2.56)$$

where the right-hand side denotes the operator dual to $\mathcal{T}(g)$. It is easy to verify that \mathcal{V} is a homomorphism on G . Furthermore, if \mathcal{T} is a continuous representation, \mathcal{V} will also be a continuous representation with respect to the weak* topology τ^\times in Φ^\times , and if \mathcal{T} is differentiable, \mathcal{V} will also be differentiable. To see this, consider a one-parameter subgroup $\{e^{tX}\}$ in G and its representation $\mathcal{T}(t, X)$ in Φ . As in (2.55), let us denote by $\mathcal{V}(t)$ the one-parameter subgroup dual to $\mathcal{T}(t, X)$. If \mathcal{T} is a differentiable representation, then for all $\phi \in \Phi$, $\lim_{t \rightarrow 0} \frac{(\mathcal{T}(t, X) - I)}{t} \phi = X\phi$, and thus,

$$\begin{aligned} \langle X\phi | F \rangle &= \left\langle \lim_{t \rightarrow 0} \frac{\mathcal{T}(t, X) - I}{t} \phi, F \right\rangle \\ &= \lim_{t \rightarrow 0} \left\langle \frac{\mathcal{T}(t, X) - I}{t} \phi, F \right\rangle \\ &= \lim_{t \rightarrow 0} \left\langle \phi, \frac{\mathcal{V}(-t) - I}{t} F \right\rangle \\ &= - \lim_{t \rightarrow 0} \left\langle \phi, \frac{\mathcal{V}(t) - I}{t} F \right\rangle \end{aligned} \quad (2.57)$$

where the second equality follows from the continuity of F as an antilinear functional on Φ .

The last equality in (2.57) shows that the $\lim_{t \rightarrow 0} \frac{\mathcal{V}(t) - I}{t} F$ exists everywhere in Φ^\times with respect to the weak* topology τ^\times . That is, the dual representation \mathcal{V} , defined by (2.55), is differentiable in Φ^\times when \mathcal{T} is differentiable in Φ . Further, since the operator X^\times dual to X is defined by $\langle X\phi, F \rangle = \langle \phi, X^\times F \rangle$, $\phi \in \Phi$, $F \in \Phi^\times$, we see from (2.57) that the generator of $\mathcal{V}(t)$ is $-X^\times$, and we may thus denote the one-parameter subgroup by $\mathcal{V}(t, -X^\times)$. It is evident that the Φ^\times -differential of \mathcal{V} , evaluated at the identity element of G , furnishes a representation V of the Lie algebra \mathcal{G} , given explicitly by

$$V(x) = -(T(x))^\times \quad x \in \mathcal{G} \quad (2.58)$$

where the $^\times$ on the right-hand side denotes the dual operator to $T(x)$. It is trivial to verify that the mapping $\mathcal{G} \rightarrow V(\mathcal{G})$ preserves the commutation relations $[x_i, x_j] = c_{ijk}x_k$ in \mathcal{G} .

2.4. Example

Proposition 2.1 shows that in a suitably constructed rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$, the restriction \mathcal{T}_Φ of a continuous Lie group representation \mathcal{T} in \mathcal{H} furnishes a differentiable representation of the group in Φ . As seen in the previous section, by duality, there also exists a

differentiable representation of the group in the dual space Φ^\times , given in particular by $(\mathcal{T}(G))^\times$. It is interesting to ask if every differentiable Lie group representation in Φ necessarily arises as the restriction of a continuous representation of the group in the kernel Hilbert space \mathcal{H} , or equivalently, if every differentiable representation in Φ extends to a continuous representation in \mathcal{H} . In this section we will construct a variant of the example considered in section 2.2 that shows that a differentiable representation in the space Φ of an RHS need not extend to a continuous representation in the Hilbert space \mathcal{H} . However, this still leaves the case for nuclear spaces unanswered because our Φ here is not a nuclear vector space.

Consider again the Lie algebra \mathcal{G} spanned by the χ_1, χ_2 and χ_3 of (2.32). The corresponding Lie group G is generated by the exponentiation of \mathcal{G} as in (2.37). We can obtain a representation of \mathcal{G} in the Hilbert space $\ell_2(\mathbb{C})$ of square summable complex sequences $\phi = (\phi_1, \phi_2, \phi_3, \dots)$ by the direct sum of the operator algebra (2.36):

$$X_1 = \sum_{n=1}^{\infty} \oplus n \chi_1 \quad X_2 = \sum_{n=1}^{\infty} \oplus n \chi_2 \quad X_3 = \sum_{n=1}^{\infty} \oplus n^2 \chi_3 \tag{2.59}$$

i.e., $X_1 \phi = (\phi_2, 0, 0, 2\phi_5, 0, 0, 3\phi_8, 0, \dots)$, etc.

The operators (2.59) are unbounded on $\ell_2(\mathbb{C})$. As a common invariant dense domain for the X_i , and therewith for the whole operator Lie algebra, we choose the subspace of rapidly decreasing sequences, $\mathcal{S} = \{\phi: \phi \in \ell_2(\mathbb{C}); \lim_{|m| \rightarrow \infty} m^n \phi_m = 0 \text{ for } n = 0, 1, 2, \dots\}$.

To obtain an RHS, we introduce on \mathcal{S} a locally convex topology by means of the scalar products

$$(\phi, \psi)_{n+1} = \sum_{i=1}^3 (X_i \phi, X_i \psi)_n + (\phi, \psi)_n \tag{2.60}$$

where $\phi, \psi \in \mathcal{S}$ and $(\phi, \psi)_0 = (\phi, \psi) = \sum_{m=1}^{\infty} \phi_m \bar{\psi}_m$, the inner product in $\ell_2(\mathbb{C})$. The ensuing norms are

$$\|\phi\|_{n+1}^2 = \sum_{i=1}^3 \|X_i \phi\|_n^2 + \|\phi\|_n^2. \tag{2.61}$$

However, from (2.35) and the definition (2.59) of the X_i , we have

$$X_i X_j = \delta_{1i} \delta_{2j} X_3. \tag{2.62}$$

Thus, the set of norms (2.61) consists of only two elements:

$$\begin{aligned} \|\phi\|_0^2 &= \|\phi\|^2 = \sum_{m=1}^{\infty} |\phi_m|^2 \\ \|\phi\|_1^2 &= \sum_{i=1}^3 \|X_i \phi\|^2 + \|\phi\|^2. \end{aligned} \tag{2.63}$$

The Hilbert space \mathcal{H}_1 which results from the completion of \mathcal{S} under the norm $\|\cdot\|_1$, its dual \mathcal{H}_1^\times and $\ell_2(\mathbb{C})$ form the RHS

$$\Phi \equiv \mathcal{H}_1 \subset \ell_2(\mathbb{C}) \subset \mathcal{H}_1^\times \equiv \Phi^\times. \tag{2.64}$$

As mentioned earlier Φ , being an infinite-dimensional Hilbert space, is not nuclear.

In much the same way as the Lie algebra of (2.33) integrates in \mathbb{R}^3 to a representation of the group G of (2.31), the operator Lie algebra spanned by the (2.59) integrates in Φ to a differentiable representation of G :

$$\mathcal{T}(\xi_1, \xi_2, \xi_3) = I + \xi_1 X_1 + \xi_2 X_2 + \xi_3 X_3. \tag{2.65}$$

That (2.65) is a homomorphism on G follows easily from (2.62) and (2.31). The continuity of $\mathcal{T}(\xi_1, \xi_2, \xi_3)$ as an operator in Φ for each $(\xi_1, \xi_2, \xi_3) \in G$, as well as the differentiability of the mapping $G \rightarrow \mathcal{T}(G)$ in $L(\Phi)$, follows from the continuity of the operators X_i and the defining relations (2.65).

Since the X_i are not continuous in $\ell_2(\mathbb{C})$, (2.65) does not yield a continuous representation of G in the central Hilbert space $\ell_2(\mathbb{C})$ of the triad (2.64). That is, the differentiable representation (2.65) in Φ does not extend to a continuous representation in $\ell_2(\mathbb{C})$. In fact, the operator Lie algebra spanned by the $\{X_i\}$ of (2.59) cannot be the differential of any continuous representation of G in $\ell_2(\mathbb{C})$, be it in the form (2.65) or not, because none of the basis elements X_i is integrable in $\ell_2(\mathbb{C})$. To see this, first notice that on the common invariant domain \mathcal{S} for the X_i ,

$$\frac{1}{\lambda^2}(\lambda + X_i)(\lambda - X_i) = \frac{1}{\lambda^2}(\lambda - X_i)(\lambda + X_i) = I \quad \lambda \neq 0. \quad (2.66)$$

If the resolvent operator $R(\lambda, X_i)$ exists for some non-zero complex number λ , it must coincide with $\frac{1}{\lambda^2}(\lambda + X_i)$ on \mathcal{S} . And for $\lambda = 0$, the range of $(\lambda - X_i)$ is not dense in $\ell_2(\mathbb{C})$. Therefore, the resolvent set of any of the X_i is empty, and the Hille–Yosida theory renders each X_i non-integrable in $\ell_2(\mathbb{C})$ to a C_0 -group.

3. Integrability of operator Lie algebras in the RHS

Example 2.4 motivates us to consider representations of Lie groups in Φ independently of possible corresponding representations of the group in \mathcal{H} .

Therefore, let us suppose that T is a representation of a d -dimensional ($d < \infty$) Lie algebra \mathcal{G} in a complex Hilbert space \mathcal{H} by linear operators defined over a common, invariant dense domain \mathcal{D} . Unlike in section 2, here we do not assume at the outset that T is the differential $d\mathcal{T}$ of a continuous Lie group representation \mathcal{T} in \mathcal{H} .

If $\{x_i\}_{i=1}^d$ is some basis for \mathcal{G} , then $T(x_i)$ furnishes a basis for $T(\mathcal{G})$, which is a finite-dimensional subspace of the algebra of endomorphisms on \mathcal{D} . We shall adopt the notation $X = T(x)$, $x \in \mathcal{G}$. Then, as in (2.2) and (2.3), we may use the operator algebra spanned by $\{X_i\}_{i=1}^d$ to define a locally convex topology on \mathcal{D} leading to an RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, where Φ is the completion of \mathcal{D} under the new locally convex topology. By construction, every X in $T(\mathcal{G})$ is continuous as an operator on Φ . Thus, the mapping T furnishes a representation of \mathcal{G} by continuous operators on Φ . We shall denote this operator Lie algebra in Φ also by $T(\mathcal{G})$, unless there is room for confusion. The problem we investigate in this section is the integrability of T :

Definition 3.1. *Let G be a connected and simply connected Lie group and \mathcal{G} its Lie algebra. Let T be a representation of \mathcal{G} by (not necessarily continuous) linear operators on a complex, locally convex, complete topological vector space Ψ . T is said to be integrable if there exists a representation \mathcal{T} of G such that its differential $d\mathcal{T}$, evaluated at the identity, contains T .*

In other words, integrability of T means that for every $x \in \mathcal{G}$, the operator $X = T(x)$ coincides on its domain of definition with the generator of the one-parameter subgroup $\mathcal{T}(e^{tx})$, $t \in \mathbb{R}$. The representation \mathcal{T} is generally taken to be continuous (definition 2.1). The well known classical results [14, 17, 18] then affirm that the generator of the one-parameter group $\mathcal{T}(e^{tx})$ is a densely defined, closed operator in Ψ . When these generators are continuous, as seen in section 2, the group representation \mathcal{T} is not simply continuous but differentiable (definition 2.2).

The integrability in the sense of definition 3.1 can be viewed as an operator valued version of Cartan's classic theorem that every abstract Lie algebra is in fact the infinitesimal Lie algebra

of a Lie group. Integrating operator Lie algebras has been a subject of continued interest [15, 16, 19–22]. Among the earlier works are that of Nelson [15] and of Flato *et al* [16], where primarily the integration of operator Lie algebras into unitary group representations in Hilbert spaces is investigated. The problem is also studied for more general cases such as Banach spaces and other locally convex spaces [19–22]. Some of these developments make use of a good deal of geometric notions, whereas [15] and [16] mainly employ techniques of functional analysis. Since the locally convex spaces in rigged Hilbert spaces have a particular topological structure as the projective limit of a scale of Hilbert spaces (2.2), for the purposes of this paper what is mostly relevant is the constructions in [15] and, in particular, [16] for Hilbert spaces; our main technical result (theorem 3.1) is an immediate extension of [16]. Therefore, we shall not review here in detail the treatments of [19–22] which deal with various aspects of the integrability problem in Banach and other locally convex spaces.

The centrally significant theoretical feature for the unitary representations is the existence of a large class of analytic vectors for the representation $T(\mathcal{G})$. In particular, Nelson proved [15] that if the Laplacian $\Delta = -\sum_{i=1}^d X_i^2$ with respect to some basis $\{X_i\}$ of $T(\mathcal{G})$ is essentially self-adjoint for a Lie algebra representation T by skew symmetric operators defined on a common invariant dense domain in a Hilbert space, then T is integrable to a unique unitary representation of G . A generalization of Nelson's integrability criterion for unitary representations was achieved by Flato *et al* (FSSS). They proved [16] that a Lie algebra isomorphism by skew symmetric operators in a Hilbert space is integrable to a unique unitary representation of G if there exists an invariant common dense domain of vectors analytic for some basis $\{X_i\}$ of the operator Lie algebra. That is, these vectors are assumed to be analytic for each X_i separately, but not necessarily for the whole Lie algebra. Thus, the FSSS theory provides less stringent integrability condition than Nelson's.

Furthermore, the FSSS theory has the interesting feature that it can be naturally generalized to continuous group representations in more general, complete locally convex spaces [16]. This generalization is achieved, however, contingent to the assumption, which supplements the ones on the existence of analytic vectors, that the closure of each basis element X_i generates a one-parameter subgroup. Although this requirement is redundant for skew symmetric operators in Hilbert spaces, the integrability problem for an operator in a general locally convex space into a continuous one-parameter group is considerably more complex, especially when the group is not globally equicontinuous in the parameter. Such was the case considered in example 2.2.

In this section, we propose an adaptation of the FSSS theory for Lie group representations in rigged Hilbert spaces. As mentioned above, for our purposes, the FSSS theory provides the most convenient and immediate starting point. Suppose then an RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$ has been built so as to yield an isomorphism T of a Lie algebra by continuous linear operators in Φ . Thus, the integrability of T amounts to finding a true anti-derivative for T , i.e., a group representation \mathcal{T} such that $d\mathcal{T} = T$ everywhere in Φ . That is, the group representation \mathcal{T} is differentiable, not just continuous as considered in [16]. Our main technical result is that the differentiability of \mathcal{T} allows us to remove the assumption on the existence of analytic vectors in the FSSS theory. This absence of the need for analytic vectors may make matters considerably simpler in applications.

Theorem 3.1. *Let $\Phi \subset \mathcal{H} \subset \Phi^\times$ be a rigged Hilbert space and $L(\Phi)$, the space of continuous linear operators in Φ equipped with the strong operator topology. Let \mathcal{G} be a Lie algebra of dimension $d < \infty$ and G the connected and simply connected Lie group with \mathcal{G} as its Lie algebra. Suppose $T: \mathcal{G} \rightarrow T(\mathcal{G}) \subset L(\Phi)$ is an isomorphism on \mathcal{G} , and suppose that there exists a basis $\{x_i\}_{i=1}^d$ for \mathcal{G} such that each $X_i \equiv T(x_i)$, $i = 1, 2, 3, \dots, d$, generates a one-parameter group in Φ . Then T is integrable to a unique differentiable representation of G .*

Before we present the proof of theorem 3.1, we shall consider some preliminary facts and identities from Lie group theory and formulate their operator valued analogues in $L(\Phi)$.

3.1. Lie algebra preliminaries

Let G , \mathcal{G} and $\{x_i\}$ be as defined in theorem 3.1. Then, a convex neighbourhood W of the identity e of G can be chosen such that any $g \in W$ can be written as

$$g = e^{t_1(g)x_1} e^{t_2(g)x_2} \dots e^{t_d(g)x_d}. \quad (3.1)$$

The coordinate functions of the second kind

$$g \rightarrow (t_1(g), t_2(g), \dots, t_d(g)) \quad (3.2)$$

furnish a local chart over W . Since W is chosen to be convex, we have $e^{tx}e^y \in W$ whenever $e^y \in W$, $e^x e^y \in W$ and $0 \leq t \leq 1$. Thus, for any $e^x \in W$ and $0 \leq t \leq 1$,

$$e^{tx} = e^{t_1(t)x_1} e^{t_2(t)x_2} \dots e^{t_d(t)x_d} \quad (3.3)$$

where we use the simpler notation $t_i(e^{tx}) \rightarrow t_i(t)$.

By applying the chain rule of differentiation on (3.3), we obtain the Lie algebra identities

$$\begin{aligned} x &= \frac{dt_1}{dt} x_1 + \dots + \frac{dt_d}{dt} \text{Int}(t_1 x_1) \dots \text{Int}(t_{d-1} x_{d-1}) x_d \\ x &= \text{Int}(-t_d x_d) \dots \text{Int}(-t_2 x_2) x_1 \frac{dt_1}{dt} + \dots + x_d \frac{dt_d}{dt} \end{aligned} \quad (3.4)$$

where $\text{Int}(tx)y = e^{tx} y e^{-tx}$, the inner automorphism on \mathcal{G} induced by the elements of G . We shall also make use of the well-known formula

$$e^{tx} y e^{-tx} = \text{Int}(tx)y = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tx))^n y \quad (3.5)$$

where $(\text{ad}(x))^n y = [x, (\text{ad}(x))^{n-1} y]$ and $(\text{ad}(x))^0 y = y$. The series on the right-hand side of (3.5) converges in the usual Euclidean topology of \mathcal{G} .

Finally, for some $x \in \mathcal{G}$ and $g \in W$ such that $e^x g \in W$, we have by the convexity of W , $e^{tx} g \in W$ for $0 \leq t \leq 1$. Thus, by (3.3),

$$e^{tx} g = e^{\alpha_1(t)x_1} e^{\alpha_2(t)x_2} \dots e^{\alpha_d(t)x_d} \quad (3.6)$$

where the α_i are analytic in t as they are simply given by the coordinate functions t_i of (3.3) as $\alpha_i(t) = t_i(e^{tx} g)$. This yields the identities [16],

$$\begin{aligned} x &= \frac{d\alpha_1}{dt} x_1 + \dots + \frac{d\alpha_d}{dt} \text{Int}(\alpha_1 x_1) \dots \text{Int}(\alpha_{d-1} x_{d-1}) x_d \\ g^{-1} x g &= \text{Int}(-\alpha_d x_d) \dots \text{Int}(-\alpha_2 x_2) x_1 \frac{d\alpha_1}{dt} + \dots + x_d \frac{d\alpha_d}{dt}. \end{aligned} \quad (3.7)$$

3.2. $L(\Phi)$ analogues

Let us denote the image of $\text{Int}(x)y$ under the isomorphism T by $\text{Int}(X)Y$, i.e., $\text{Int}(X)Y \equiv T(\text{Int}(x)y)$.

Proposition 3.1.

$$(\text{Int}(tX)Y)\phi = T(e^{tx} y e^{-tx})\phi = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tX))^n Y\phi \quad \phi \in \Phi \quad (3.8)$$

where $(\text{ad}(X))^n Y\phi = X(\text{ad}(X))^{n-1} Y\phi - (\text{ad}(X))^{n-1} YX\phi$.

Proof. The first equality follows trivially from the first equality of (3.5). What needs to be shown is the convergence of the series in $L(\Phi)$ and that its limit is $\text{Int}(tX)Y$. But this is trivial from (3.5) and the continuity of the mapping $T: \mathcal{G} \rightarrow L(\Phi)$ in the strong operator topology of $L(\Phi)$. \square

Recall that the basis $\{x_i\}$ is chosen in \mathcal{G} so that each $X_i = T(x_i)$ integrates to a one-parameter group of operators in Φ . Let $\mathcal{T}(t, X_i)$ be this group. Then, by the continuity of X_i ,

$$\frac{d}{dt}\mathcal{T}(t, X_i)\phi = X_i\mathcal{T}(t, X_i)\phi = \mathcal{T}(t, X_i)X_i\phi \quad \phi \in \Phi. \quad (3.9)$$

Further, since Φ is a Fréchet space, $\mathcal{T}(t, X_i)$ is locally equicontinuous, i.e., for any compact interval $I \subset \mathbb{R}$ and any n , there exists some m such that

$$\|\mathcal{T}(t, X_i)\phi\|_n \leq \|\phi\|_m \quad t \in I \quad \phi \in \Phi. \quad (3.10)$$

The relations (3.9) and (3.10) are among the standard results of the theory of one-parameter groups in locally convex spaces [14].

Proposition 3.2.

$$\frac{d}{dt}\mathcal{T}(t, X_i)\mathcal{T}(t, X_j)\phi = \mathcal{T}(t, X_i)(X_i + X_j)\mathcal{T}(t, X_j)\phi \quad \phi \in \Phi. \quad (3.11)$$

Proof.

$$\begin{aligned} & \left\| \frac{d}{dt}\mathcal{T}(t, X_i)\mathcal{T}(t, X_j)\phi - \mathcal{T}(t, X_i)(X_i + X_j)\mathcal{T}(t, X_j)\phi \right\|_n \\ &= \lim_{s \rightarrow 0} \left\| \frac{\mathcal{T}(t+s, X_i)\mathcal{T}(t+s, X_j) - \mathcal{T}(t, X_i)\mathcal{T}(t, X_j)}{s} \phi \right. \\ & \quad \left. - \mathcal{T}(t, X_i)(X_i + X_j)\mathcal{T}(t, X_j)\phi \right\|_n \\ & \leq \lim_{s \rightarrow 0} \left\| \mathcal{T}(t+s, X_i) \frac{\mathcal{T}(t+s, X_j) - \mathcal{T}(t, X_j)}{s} - \mathcal{T}(t+s, X_i)X_j\mathcal{T}(t, X_j)\phi \right\|_n \\ & \quad + \lim_{s \rightarrow 0} \left\| \mathcal{T}(t+s, X_i)X_j\mathcal{T}(t, X_j)\phi - \mathcal{T}(t, X_i)X_j\mathcal{T}(t, X_j)\phi \right\|_n \\ & \quad + \lim_{s \rightarrow 0} \left\| \frac{\mathcal{T}(t+s, X_i) - \mathcal{T}(t, X_i)}{s} \mathcal{T}(t, X_j)\phi - \mathcal{T}(t, X_i)X_i\mathcal{T}(t, X_j)\phi \right\|_n \\ & \leq \lim_{s \rightarrow 0} \left\| \frac{\mathcal{T}(t+s, X_j) - \mathcal{T}(t, X_j)}{s} - X_j\mathcal{T}(t, X_j)\phi \right\|_m \\ & \quad + \lim_{s \rightarrow 0} \left\| \mathcal{T}(t+s, X_i)X_j\mathcal{T}(t, X_j)\phi - \mathcal{T}(t, X_i)X_j\mathcal{T}(t, X_j)\phi \right\|_n \\ & \quad + \lim_{s \rightarrow 0} \left\| \frac{\mathcal{T}(t+s, X_i) - \mathcal{T}(t, X_i)}{s} \mathcal{T}(t, X_j)\phi - \mathcal{T}(t, X_i)X_i\mathcal{T}(t, X_j)\phi \right\|_n. \end{aligned}$$

The first term in the last inequality follows from the local equicontinuity of \mathcal{T} , (3.10). Since each term on right-hand side tends to zero, we have (3.11). Notice that we needed only the local equicontinuity of $\mathcal{T}(t, X_i)$ but not that of $\mathcal{T}(t, X_j)$ for (3.11) to hold. \square

Relations (3.8) and (3.11) can be combined to obtain an $L(\Phi)$ analogue of the Lie algebra identity (3.5):

Proposition 3.3. For any two basis elements X_i and X_j of $T(\mathcal{G})$, the equality

$$\mathcal{T}(t, X_i)X_j\mathcal{T}(-t, X_i)\phi = \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tX_i))^n X_j\phi \quad (3.12)$$

holds for all $\phi \in \Phi$. The series here is defined as in (3.8) in the strong operator topology.

Proof. The proposition is clearly true for $t = 0$. Next, by the continuity of the linear mapping $T: \mathcal{G} \rightarrow L(\Phi)$, we have

$$\frac{d}{dt} T(e^{tX_i}x_j e^{-tX_i})\phi = T\left(\frac{d}{dt}(e^{tX_i}x_j e^{-tX_i})\right)\phi = T(e^{tX_i}(\text{ad}(X_i)x_j)e^{-tX_i})\phi. \quad (3.13)$$

But, by (3.5),

$$e^{tX_i}(\text{ad}(X_i)x_j)e^{-tX_i} = (\text{ad}(X_i))(e^{tX_i}x_j e^{-tX_i}) = (\text{ad}(X_i)) \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tX_i))^n x_j. \quad (3.14)$$

Thus, again by the continuity of $T: \mathcal{G} \rightarrow L(\Phi)$,

$$T((\text{ad}(X_i))(e^{tX_i}x_j e^{-tX_i}))\phi = (\text{ad}(X_i)) \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tX_i))^n X_j\phi. \quad (3.15)$$

Therefore, from (3.8), (3.13) and (3.15), we have

$$\frac{d}{dt} \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tX_i))^n X_j\phi = (\text{ad}(X_i)) \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tX_i))^n X_j\phi \quad \phi \in \Phi. \quad (3.16)$$

Now, from (3.11),

$$\begin{aligned} \frac{d}{dt} \mathcal{T}(t, X_i)X_j\mathcal{T}(-t, X_i)\phi &= \mathcal{T}(t, X_i)(\text{ad}(X_i)X_j)\mathcal{T}(-t, X_i)\phi \\ &= (\text{ad}(X_i))(\mathcal{T}(t, X_i)X_j\mathcal{T}(-t, X_i))\phi \quad \phi \in \Phi. \end{aligned} \quad (3.17)$$

Equalities (3.16) and (3.17) yield the $L(\Phi)$ valued differential equation

$$\frac{d}{dt} u(t)\phi = (\text{ad}(X_i))u(t)\phi \quad \phi \in \Phi \quad (3.18)$$

where

$$u(t) = \mathcal{T}(t, X_i)X_j\mathcal{T}(t, X_i) - \sum_{n=0}^{\infty} \frac{1}{n!} (\text{ad}(tX_i))^n X_j. \quad (3.19)$$

Thus $u(0) = 0$. We can employ a technique used in [16], equation (12), redefined here with respect to the $L(\Phi)$ topology, to show that the solution $u(t)$ to (3.18) is identically equal to zero. To that end, consider the function $v(s)\phi = \mathcal{T}(t-s, X_i)u(s)\mathcal{T}(-t+s, X_i)\phi$, where $u(s)$ is as in (3.19).

From (3.8) and (3.10), $u(s)$ is locally equicontinuous in s . Hence, by proposition 3.2,

$$\frac{dv(s)}{ds}\phi = -X_i v(s)\phi + (\text{ad}(X_i))v(s)\phi + v(s)X_i\phi = 0 \quad (3.20)$$

i.e., $v(s)$ is independent of s . Therefore,

$$u(t) = v(t) = v(0) = \mathcal{T}(t, X_i)u(0)\mathcal{T}(-t, X_i) = 0 \quad (3.21)$$

and (3.19) gives (3.12). \square

In summary,

$$\mathcal{T}(t, X_i)X_j\mathcal{T}(-t, X_i)\phi = (\text{Int}(tX_i)X_j)\phi = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\text{ad}(X_i))^n X_j\phi. \quad (3.22)$$

Remark. This equality is similar to equation (8) of [16]. However, our assumptions as well as proof technique are different.

Further, from (3.4) and (3.7), we also have the $L(\Phi)$ valued the Lie algebra identities:

$$\begin{aligned} X &= \frac{dt_1}{dt}X_1 + \cdots + \frac{dt_d}{dt}\text{Int}(t_1X_1) \cdots \text{Int}(t_{d-1}X_{d-1})X_d \\ X &= \text{Int}(-t_dX_d) \cdots \text{Int}(-t_2X_2)X_1 \frac{dt_1}{dt} + \cdots + X_d \frac{dt_d}{dt} \\ X &= \frac{d\alpha_1}{dt}X_1 + \cdots + \frac{d\alpha_d}{dt}\text{Int}(\alpha_1X_1) \cdots \text{Int}(\alpha_{d-1}X_{d-1})X_d. \end{aligned} \quad (3.23)$$

All the technical preliminaries are now in place for the proof of theorem 3.1.

3.3. Proof of theorem 3.1

Let W be as defined in section 3.1. Then, for any $g \in W$ (i.e., of the form (3.1)) we define an $L(\Phi)$ element $\mathcal{T}(g)$ by

$$\mathcal{T}(g) = \mathcal{T}(t_1(g), X_1)\mathcal{T}(t_2(g), X_2) \cdots \mathcal{T}(t_d(g), X_d). \quad (3.24)$$

Being the composition of finitely many continuous linear operators, $\mathcal{T}(g)$ is a continuous linear operator. Next, if $x \in \mathcal{G}$ is such that $e^x \in W$, then for $0 \leq t \leq 1$ by way of (3.3) and (3.6),

$$\begin{aligned} \mathcal{T}(e^{tx}) &= \mathcal{T}(t_1(t), X_1)\mathcal{T}(t_2(t), X_2) \cdots \mathcal{T}(t_d(t), X_d) \\ \mathcal{T}(e^{tx}g) &= \mathcal{T}(\alpha_1(t), X_1)\mathcal{T}(\alpha_2(t), X_2) \cdots \mathcal{T}(\alpha_d(t), X_d). \end{aligned} \quad (3.25)$$

Since each $\mathcal{T}(t, X_i)$ is locally equicontinuous, repeated applications of proposition 3.2 and proposition 3.3 ((3.22) in particular) on the first equality in (3.25) yield, for all $\phi \in \Phi$,

$$\begin{aligned} \frac{d}{dt}\mathcal{T}(e^{tx})\phi &= \left(\frac{dt_1}{dt}X_1 + \cdots + \frac{dt_d}{dt}\text{Int}(t_1X_1) \cdots \text{Int}(t_{d-1}X_{d-1})X_d \right) \mathcal{T}(e^{tx})\phi \\ \frac{d}{dt}\mathcal{T}(e^{tx}g)\phi &= \mathcal{T}(e^{tx}) \left(\text{Int}(-t_dX_d) \cdots \text{Int}(-t_2X_2)X_1 \frac{dt_1}{dt} + \cdots + X_d \frac{dt_d}{dt} \right) \phi. \end{aligned}$$

Thus, with (3.23), we have, for all $\phi \in \Phi$,

$$\frac{d}{dt}\mathcal{T}(e^{tx})\phi = X\mathcal{T}(e^{tx})\phi = \mathcal{T}(e^{tx})X\phi \quad \phi \in \Phi. \quad (3.26)$$

This shows the differentiability of $\mathcal{T}(e^{tx})$ in the neighbourhood W of the identity of G .

The same application on the second equality in (3.25), together with (3.23), gives

$$\frac{d}{dt}\mathcal{T}(e^{tx}g)\phi = X\mathcal{T}(e^{tx}g)\phi \quad \phi \in \Phi \quad (3.27)$$

Next, for $0 \leq s \leq t \leq 1$, the vector valued function

$$f(s)\phi = \mathcal{T}(e^{sx})\mathcal{T}(e^{(t-s)x}g)\phi \quad (3.28)$$

can be differentiated, as in (3.11), because $\mathcal{T}(e^{sx})$ and $\mathcal{T}(e^{(t-s)x}g)$ are both locally equicontinuous in s . Thus,

$$\frac{d}{ds}f(s)\phi = X\mathcal{T}(e^{sx})\mathcal{T}(e^{(t-s)x}g)\phi - \mathcal{T}(e^{sx})X\mathcal{T}(e^{(t-s)x}g)\phi = 0 \quad (3.29)$$

That is, $f(s)\phi$ is independent of s , and so,

$$f(0)\phi = T(e^{tx})\phi = f(t)\phi = T(e^{tx})T(g)\phi. \quad (3.30)$$

This shows that the mapping $W \rightarrow T(g)$ defined by (3.24) is a homomorphism on W .

Recall that G was assumed to be the connected and simply connected Lie group with \mathcal{G} as its Lie algebra. Thus, an arbitrary element g of G can be written as a product of finitely many elements of W . Consequently, the homomorphism $T: W \rightarrow L(\Phi)$ given by (3.24) can be extended from W to the entire group manifold, and the simply connectedness of G assures that this extension is well defined for all $g \in G$. From (3.26) and the analyticity of the multiplication in G , it follows that the above extension yields a differentiable representation of G in Φ . It is straightforward to verify, by way of (3.27), that the differential $dT|_e$ coincides with the Lie algebra representation given at the outset, T .

This concludes the proof of theorem 3.1. \square

As an immediate consequence of the theorem, we have the following corollary:

Corollary 3.1. *Under the assumptions of theorem 3.1, the dual Lie algebra representation in Φ^\times , defined by $T^\times(x) = -(T(x))^\times$, $x \in \mathcal{G}$, is integrable.*

Proof. If T is integrable to the differentiable representation \mathcal{T} in Φ , then as defined by (2.56), there exists a differentiable representation \mathcal{V} in Φ^\times . The weak* differential $d\mathcal{V}$ of \mathcal{V} is precisely $-(T(\mathcal{G}))^\times$. \square

Example 2.4 led us to the conclusion that not every differentiable Lie group representation in Φ comes about as the restriction of a continuous representation of the group in \mathcal{H} . The following proposition allows us to determine if such is the case for a given differentiable group representation in Φ of an RHS.

Proposition 3.4. *Let \mathcal{G} and G be as in theorem 3.1, and let \mathcal{T} be a differentiable representation of G in the space Φ of a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$. Suppose there exists a basis $\{X_i\}_{i=1}^d$ for the operator Lie algebra $T(\mathcal{G})$ such that each one-parameter subgroup $\mathcal{T}(t, X_i)$ extends to a continuous one-parameter subgroup in \mathcal{H} . Then the differentiable representation \mathcal{T} extends to a continuous representation of G in \mathcal{H} .*

Proof. Since the extension of the one-parameter subgroup $\mathcal{T}(t, X_i)$ in \mathcal{H} is generated by the \mathcal{H} -closure \bar{X}_i of the generator X_i , let us denote it by $\mathcal{T}(t, \bar{X}_i)$. Now, for $g \in W$, where W is as in the proof of theorem 3.1, define

$$\mathcal{T}_{\mathcal{H}}(g)\phi = \mathcal{T}(t_1(g), \bar{X}_1) \cdots \mathcal{T}(t_d(g), \bar{X}_d)\phi \quad \phi \in \mathcal{H} \quad g \in W. \quad (3.31)$$

It is clear that $\mathcal{T}_{\mathcal{H}}(g)$ is a continuous linear operator in \mathcal{H} for each $g \in W$. Since $\mathcal{T}_{\mathcal{H}}(g)$ coincides with $\mathcal{T}(g)$ of (3.24) on Φ and since Φ is dense in \mathcal{H} , the mapping $\mathcal{T}_{\mathcal{H}}: W \rightarrow \mathcal{B}(\mathcal{H})$ of (3.31) is a homomorphism on W . For $x \in \mathcal{G}$ such that $e^{tx} \in W$, $0 \leq t \leq 1$, we have

$$\mathcal{T}_{\mathcal{H}}(e^{tx})\phi = \mathcal{T}(t_1(t), \bar{X}_1)\mathcal{T}(t_2(t), \bar{X}_2) \cdots \mathcal{T}(t_d(t), \bar{X}_d)\phi \quad \phi \in \mathcal{H} \quad (3.32)$$

which shows that $\mathcal{T}_{\mathcal{H}}: W \rightarrow \mathcal{B}(\mathcal{H})$ is continuous on W . As in the proof of theorem 3.1, the connectedness and simply connectedness of G permits a well defined extension of $\mathcal{T}_{\mathcal{H}}$ from W to the entire G to yield a continuous representation of G in \mathcal{H} . \square

In view of proposition 2.1, \mathcal{T} in Φ is then the projective limit of continuous representations of G in a scale of Hilbert spaces $\mathcal{H} \supset \mathcal{H}_1 \supset \mathcal{H}_2 \cdots$.

4. Concluding remarks

This paper studies some aspects of differentiable representations of finite-dimensional Lie groups in rigged Hilbert spaces. In particular, it is shown (proposition 2.1) that, for a suitably constructed rigged Hilbert space, such a representation can always be obtained from a continuous representation of the group defined in a Hilbert space. Further, conditions are specified (theorem 3.1) under which a given Lie algebra representation in a Hilbert space may be integrated to an RHS representation of the corresponding Lie group. It is worthwhile to point out that, in a suitable RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, such integrability may be possible in Φ even when the given Hilbert space representation of the Lie algebra is not integrable in the Hilbert space \mathcal{H} itself (proposition 3.4).

Lie groups and Lie algebras play an essential role in many quantum mechanical theories. Building a part of the theoretical framework for handling Lie group and algebra representations in the RHS formulation of quantum mechanics is the primary goal of this paper. In addition, as pointed out in the introduction, recent applications of the formalism to characterize relativistic resonances and unstable particles involve intricacies of the representations of Lie groups (and subsemigroups thereof) in RHS. In the developments achieved in [9], the space Φ , and therewith the RHS $\Phi \subset \mathcal{H} \subset \Phi^\times$, is built so that a differentiable representation of the Poincaré semigroup (introduced in section 1) can be obtained in Φ from a unitary representation of the Poincaré group in \mathcal{H} . In particular, these constructions employ proposition 2.1 to obtain a differentiable representation of the homogeneous Lorentz group in Φ . Further, the construction of Φ is done so that the momentum operators P_μ do not generate one-parameter groups in Φ , and thus (theorem 3.1) the differentiable representation of the Poincaré semigroup in Φ does not extend to a representation of the entire Poincaré group. Motivation for the mathematical developments presented in this paper partly comes from the theory of relativistic resonances and unstable particles proposed in [9].

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